Analytical and numerical research of the Celt stone dynamics

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Abstract. The problem of the Celt stone motion is investigated. The convenient equations of its small motions with the exactness of the second order are delivered. From these equations the main Celt effects may be obtained. By the averaging method the equation describing the vertical projection of the angular velocity is delivered. In contrast to the classical works here we take into account the small friction couples. We study the effect of the roll friction, of the rotating friction, and of the viscous resistance of the air.

1 Introduction

The Celt stone motion was investigated more than 100 years ago. The approximate solution which describes all Celt effects is given by A.P.Markeev [1]. The classical Celt stone is a body which moves on the rough horizontal plane under action of gravity. The body has the 3-axial inertia ellipsoid, in the equilibrium state one of the main inertia axes is vertical and it crosses the body surface in the contact point. The principal property of this body is that it is convex and smooth near the contact point, the main surface curvatures in this point are not equal to each other, and the main surface curvature directions do not coincide with the directions of the main inertia axes.

2 Equations of motion

Let the body surface near the contact point is paraboloid with the main radii of curvature $R_1, R_2 > h$ in its top. The coordinate system is connected with the body. The body (Fig. 1) moves without sliding and jumping under action of gravity and the plane reaction. The body position (see, P.A.Zhilin [2]) is given by the turn tensor *P*. We present the rotation as a composition of two turns through the Euler angles (Fig.2)

$$\underline{P} = \underline{P}(\psi \underline{k}) \bullet \underline{P}(\theta \underline{i}) \bullet \underline{P}(\varphi \underline{k}) = \underline{P}_2(\gamma \underline{k}) \bullet \underline{P}_1(\underline{\theta}), \quad \underline{\theta} = \theta \cos\varphi \underline{i} - \theta \sin\varphi \underline{j} = \theta_1 \underline{i} + \theta_2 \underline{j}, \quad \gamma = \varphi + \psi,$$

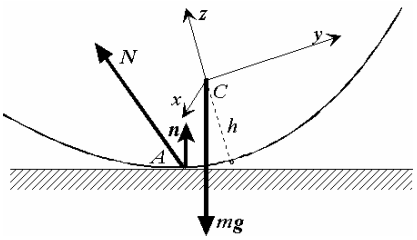


Fig. 1. The body moving

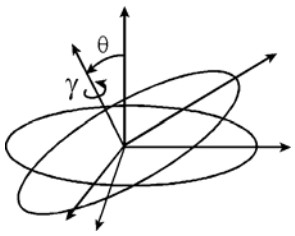


Fig. 2. The turn of the body

and find the angular velocity Ω, satisfying to the Poisson's equation

$$\underline{P}^{\bullet} = \underline{P} \times \underline{\Omega}, \qquad \underline{\Omega} = \gamma^{\bullet}\underline{k} - \underline{\theta} \times \gamma^{\bullet}\underline{k} + \underline{\theta}^{\bullet} - \tfrac{1}{2}\underline{\theta} \times \underline{\theta}^{\bullet} - \tfrac{1}{2}\theta^2\gamma^{\bullet}\underline{k} + \ldots$$

For analysis we use the approximate system which holds the terms of the second order with respect to the small angles of rotations and their derivatives

$$\begin{aligned}
&\left(A_{11} + mh^2\right)\theta_1^{\bullet\bullet} + A_{12}\theta_2^{\bullet\bullet} + mg\left(R_2 - h\right)\theta_1 = \left(A_{11} + A_{22} - A_{33} + mh\left(2h - R_1\right)\right)\Omega_3\theta_2^{\bullet}, \\
&A_{12}\theta_1^{\bullet\bullet} + \left(A_{22} + mh^2\right)\theta_2^{\bullet\bullet} + mg\left(R_1 - h\right)\theta_2 = -\left(A_{11} + A_{22} - A_{33} + mh\left(2h - R_2\right)\right)\Omega_3\theta_1^{\bullet}, \\
&A_{33}\Omega_3^{\bullet} = \left(A_{11} - A_{22}\right)\theta_1^{\bullet}\theta_2^{\bullet} + A_{12}\left(\theta_2^{\bullet 2} - \theta_1^{\bullet 2}\right) + mh\left(R_2\theta_1\theta_2^{\bullet\bullet} - R_1\theta_2\theta_1^{\bullet\bullet}\right) + mg\left(R_1 - R_2\right)\theta_1\theta_2.
\end{aligned} \qquad (1)$$

Here $\underline{A}$ is the inertia tensor in the reference position, m is the mass, Ω_3 is the angular velocity projection on the axis z, $\Omega_3 = \underline{\Omega} \bullet \underline{k}$.

The Celt stone rotation around the vertical axis is unstable for a small enough angular velocity. The rotation around the vertical axis in one direction generates vibrations around the horizontal axes and then these vibrations transform to the rotation around the vertical axis in the other direction and so on.

In the first approximation the right sides of equations (1) are equal to zero and they describe the small free vibrations which the natural frequencies ν_1 and ν_2 around the horizontal axes and rotation with the constant velocity. The terms of the second order lead to the energy transition. Further we are interesting with the slowly changed amplitudes и p and q and with the evolution of the angular velocity. By the averaging method system (1) is reduced to the approximation system of third order [1]

$$\frac{dp}{dt} = -a\nu_1^{\,2}p\Omega_3, \qquad \frac{dq}{dt} = a\nu_2^{\,2}q\Omega_3, \qquad \frac{d\Omega_3}{dt} = \frac{a}{A_{33}}\left(\nu_1^{\,4}p^2 - \nu_2^{\,4}q^2\right). \qquad (2)$$

It is known [1] that $\nu_1^2 p^2 + \nu_2^2 q^2 + A_{33}\Omega_3^{\,2} = C_1 = const, \quad p^{\chi}q = C_2 = const, \quad \chi = (\nu_1/\nu_2)^2.$ (3)

The representing point trajectory in the space p, q, Ω_3 lies on the ellipsoid (Fig.3).

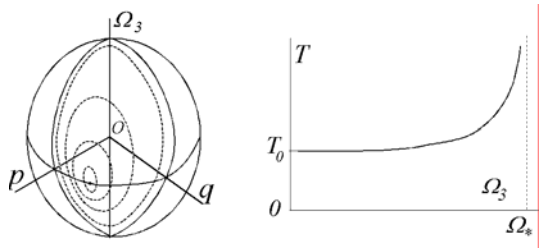
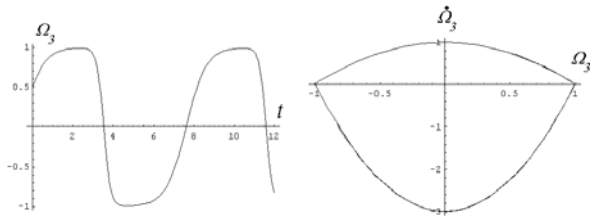


Fig. 3. The trajectories and the period

From relations (2) and (3) by introducing new dimensionless variables we obtain the equation for the angular velocity with parameter χ.

$$\ddot{\Omega}_3 + 2\left(\chi\left(1 - \Omega_3^2\right) + \left(1 - \chi\right)\dot{\Omega}_3\right)\Omega_3 = 0, \qquad \chi = (\nu_1/\nu_2)^2. \qquad (4)$$

The numerical periodic solution of this equation for $\chi = 5$ is shown in Fig. 4. The exact system solution (Fig. 6) accords with the graphics for $\Omega_3(t)$ (Fig. 4, left), but the exact value $\dot{\Omega}_3$ (Fig. 5) differs from the approximate one (Fig. 4, right).



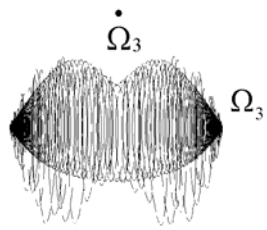


Fig. 4.The solution of the averaging equation (4) Fig. 5. The exact solution.

3 The friction effect

We expand the angular velocity by the sum of two orts $\underline{\Omega} = \Omega_n \underline{e}_n + \Omega_{pn} \underline{e}_{pn}$.

Let the entire friction moment is consist of three parts: of the rolling friction, of the rotation friction and of the viscous air resistance

$$\underline{M}_{fr} = -k_{roll} N_n \underline{e}_{pn} - k_{rot} N_n \underline{e}_n - \underline{B}_{air} \bullet \underline{\Omega}, \qquad N_n = \underline{N} \bullet \underline{n}.$$

With friction system (2) accepts the form

$$\begin{aligned} \frac{dp}{dt} &= -a\nu_1^2 p\Omega_3 - k_{air1} p - k_{roll} I_1(p,q), \\ \frac{dq}{dt} &= a\nu_2^2 q\Omega_3 - k_{air2} q - k_{roll} I_2(p,q), \\ \frac{d\Omega_3}{dt} &= \frac{a}{A_{33}}\left(\nu_1^4 p^2 - \nu_2^4 q^2\right) - k_{air3}\Omega_3 - k_{rot}\frac{\Omega_3}{|\Omega_3|}, \end{aligned} \qquad \begin{aligned} I_1 &= \lim_{T_0\to\infty}\frac{1}{T_0}\int_0^{T_0}\frac{\theta_1^{\bullet} + \alpha_1\theta_2^{\bullet}}{\nu_1 a_1 |\theta^{\bullet}|}\cos(\nu_1 t + \beta_1)\, dt, \\ I_2 &= \lim_{T_0\to\infty}\frac{1}{T_0}\int_0^{T_0}\frac{\theta_2^{\bullet} + \alpha_2\theta_1^{\bullet}}{\nu_2 a_2 |\theta^{\bullet}|}\cos(\nu_2 t + \beta_2)\, dt, \end{aligned} \qquad (5)$$

In this case values C_1, C_2 in relation (3) become slowly changing functions of time

and satisfy to equations $\dfrac{dC_1}{dt} = f_1(p,q,\Omega_3), \quad \dfrac{dC_2}{dt} = f_2(p,q,\Omega_3)$. (6)

For small friction by averaging of the right sides of equations (6) we obtain

$$\frac{dC_i}{dt} = F_i(C_1, C_2) = \langle f_i \rangle = \frac{1}{T}\int_0^T f_i(p,q,\Omega_3)\, dt, \qquad i = 1, 2. \qquad (7)$$

Equations (7) allow us to observe the trajectory evolution on the phase ellipsoid. The various types of friction effect for various initial conditions are studied numerically.

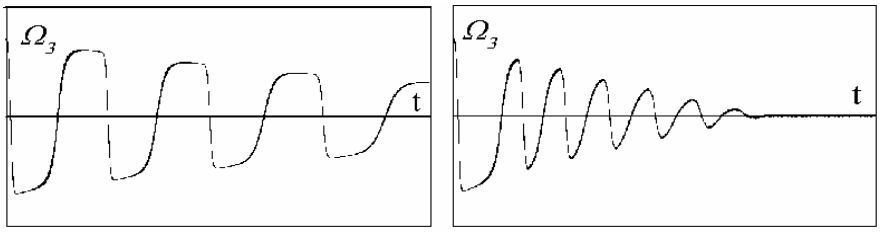


Fig. 6. The air resistance and the rotation friction effects

The rotation friction mainly acts on the angular velocity Ω_3 but fully it does not dump out the vibrations.

The rolling friction acts on the angular velocity more slowly, but it dump out vibrations and simultaneously stop the energy transition and Celt effect. The air resistance leads to the period increase.

References

1. Markeev A.P.: Dynamics of a body being contiguous to a rigid surface. Moscow, Nauka, 1992, 336p. (in Russian)
2. Zhilin P.A.: A New Approach to the Analysis of Free rotations of Rigid Bodies. ZAMM, Z.angew. Math. Mech. 76 (1996), 4, 187-204